

# Extended Gauge Symmetries in F-theory

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## Abstract

We study gauge symmetry in F-theory in light of global aspects. For this, we consider not only a simple (local) group, but also a semi-simple group with Abelian factors. Once we specify the complete gauge group by decomposing the discriminant, analogous to arranging 7-branes, we can derive the matter contents, their localization and the relation to enhanced groups. Global constraints coming from Calabi–Yau conditions and anomaly cancellations imply a unified group. The semisimple group shows explicit formation of matter curves and nontriviality of its embedding into exceptional group. Also the dual heterotic string vacua with line bundles provide a guide on the unification.

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# 1 Introduction

We study gauge symmetry in F-theory focusing on the global nature of 7-branes. F-theory [1–3] is a good unified framework where gauge symmetry is readily described by branes generalized from those of open string, as well as heterotic dual is directly established. Also exceptional group is naturally obtainable in both pictures. On the F-theory side, we have accessible tool for describing gauge group in terms of singularity of compact space of the same name, sharing the same connectedness. In the perturbative limit, the singularity is interpreted as  $(p, q)$ -branes and orientifold planes [4, 5]. The matter contents are obtained by purely geometric way, e.g. from the intersections of 7-branes. On the heterotic side, a similar brany description is translated to the information on instantons [6, 7]. On both sides, the moduli space and the spectrum are independently well-understood [8–10].

So far, much attention is paid on a unified model based on a simple group in the local picture. The decoupling limit  $M_{\text{Pl}}/M_{\text{GUT}} \rightarrow \infty$  allows us to concentrate on one local group and to make easy bottom-up construction [11–14]. On the other hand, it is also noted that the matter contents and their interactions are accounted by gauge symmetry enhancement. This is naturally explained by further unification to a larger group. The question, to what extent and what kind of enhancement is possible, leads us to seek the global structure.

On the heterotic side, a simple gauge group occurs under the instanton background as a nonabelian gauge bundle. At the generic point of moduli space, this group can be the only surviving one, hence in the sense of heterotic duality, the local picture with a single simple group is also globally consistent. However, this is not the only possibility. In the low energy limit of F-theory, we have an adjoint scalar  $\phi_{mn}$ , spatially transforming as the canonical divisor on the compact space, parameterizing the normal direction to the 7-branes [11], which is similar to the scalar for the D7-brane in the perturbative picture. A nonzero vacuum expectation value (VEV) of this field gives rise to brane separation, and a non-constant expectation value makes the branes intersecting. As a result, we can have more than one simple or Abelian subgroup. Again, on the heterotic side, a line bundle background that gives rise to semisimple and Abelian gauge group is crucial to understand. Because the line bundle commutes with every other group, the rank is not reduced, giving rise to more fruitful spectrum [15, 16]. For this, in Section 2, we extend the work of Ref. [8] to access the full decomposition of discriminant, using intersection theory. It is powerful for describing more than one group on the equal footing, contrary to the Weierstrass form that can see only one group at once in general.

We will see that there are constraints from the Calabi–Yau condition and anomaly cancellations, so that we cannot have arbitrary large group. Usually this fact indicates that we can understand at least a certain class of vacua as broken symmetry of some

unified group [17]. In Section 2, we also analyze the spectrum and moduli space and seek relations to some unification group. Also we can think about the opposite direction of obtaining a realistic model from a given unified group.

Furthermore, there are some objects and observables that are to be *derived* from top-down information [18–21]. In the local picture, the matter curve, where a matter of a certain representation is localized, is assumed to wrap on a certain subcycle on the brane. If we specify a Calabi–Yau manifold, the background on this manifold should give us sufficient information on low-energy spectrum and coupling, calculated by index theorem [11, 12]. It is done by relating geometric data to Calabi–Yau information and specifying the cycles supporting the gauge group, constrained by global consistency condition. If we then know how the gauge symmetry is broken, in principle we should be able to calculate the real location of the matter curves. This is the case in the perturbative picture, we have the clear answer that a chiral matter under the bifundamental representation is localized at the intersections of two stacks of D-branes. Generalization of such mechanism to exceptional groups is known in mathematics literatures [9, 23], however its application to physics have been limited [24–26]. We look for such top-down picture in the F-theory description, where a nontriviality comes from the exceptional gauge group. And Yukawa coupling involves the normalization of matter wavefunctions along the entire volume of the cycle they live on [21, 27, 28]. In Section 2 we consider the gauge sector, mainly in six dimension where the only necessary global consistency condition is 7-brane charge conservation [2, 3, 22]. In Section 3, we study such derived objects from the 7-branes. Many examples, especially semisimple groups, are dealt with in Section 4. We conclude in Section 5.

## 2 Gauge fields on 7-branes

### 2.1 Description

At present, the only possible definition of F-theory is via type IIB string theory. Namely, we identify the  $SL(2, \mathbb{Z})$  symmetric axion-dilaton field  $\tau$  of IIB with the complex structure of an extra torus, lifting the theory twelve-dimensional [1].<sup>1</sup> This requires  $X_4$  to be an elliptic fibration over a three-dimensional base  $B'_3$ ,  $\pi' : X_4 \rightarrow B'_3$ . With a section, the elliptic fiber admits description in terms of Weierstrass equation

$$y^2 = x^3 + fx + g, \tag{1}$$

where  $f$  and  $g$  are polynomials on  $B'_3$ . For the total space  $X_4$  to be Calabi–Yau,  $f$  and  $g$  transform as holomorphic sections of  $-4K_{B'_3}$  and  $-6K_{B'_3}$  respectively, where  $K_{B'_3}$  is

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<sup>1</sup>See Refs. [30, 31] for general introduction.

ord $f$	ord $g$	ord $\Delta$	fiber	group
$\geq 0$	$\geq 0$	0	$I_0$	$\cdot$
0	0	$n$	$I_n$	$A_{n-1}$
$\geq 1$	1	2	II	$\cdot$
1	$\geq 2$	3	III	$A_1$
$\geq 2$	2	4	IV	$A_2$
2	$\geq 3$	$n+6$	$I_n^*$	$D_{n+4}$
$\geq 2$	3	$n+6$	$I_n^*$	$D_{n+4}$
$\geq 3$	4	8	$IV^*$	$E_6$
3	$\geq 5$	9	$III^*$	$E_7$
$\geq 4$	5	10	$II^*$	$E_8$

Table 1: Kodaira classification of singularities.

the canonical class of the base manifold  $B'_3$  [29].<sup>2</sup> Roughly it means  $f$  and  $g$  can be locally viewed as function of coordinates in  $B'_3$ , globalizing with topological numbers 4 and 6, respectively. Due to the  $SU(4)$  special holonomy of  $X_4$ , the compactification leaves 1/8 of the supersymmetry, which is  $\mathcal{N} = 1$  in terms of four dimensional supersymmetry. The complex structure  $\tau$  of the elliptic (torus) fiber is related to  $f, g$  through a modular function called the  $j$ -function [1],

$$j(\tau) = \frac{4(24f)^3}{\Delta}, \quad \Delta \equiv 4f^3 + 27g^2. \quad (2)$$

The discriminant locus  $D$  is a divisor, or collection of codimension one subspaces, of  $B'_3$  specified by the equation  $\Delta = 0$ .  $\Delta$  transforms as a section of  $-12K_{B'_3} = 12c_1(B'_3)$  and is expressed as a formal sum of irreducible divisors of  $B'_3$

$$D = -12K_{B'_3} = \sum_i n_i S_i, \quad n_i \geq 0. \quad (3)$$

Later, we will check to what extent this form is meaningful.

Going close to *one*  $S_i$ , the elliptic fiber degenerates and gives rise to a singularity in the torus, in the sense of (2) with  $\Delta \rightarrow 0$ . The corresponding curve at  $S_i$  locally looks like  $y^2 = x^3 + f_i x + g_i$  and carries the orders  $\text{ord}(f_i, g_i, \Delta_i)$ , displayed in Table 1. In the relation (3),  $n_i = \text{ord} \Delta_i$  at the corresponding  $S_i$  [2]. It gives rise to the world-volume gauge group of the same name. In string theory language, each  $S_i$  provides four-cycle which a number of 7-branes wrap, which carries some units of RR and NSNS charges. In case of D-branes, only the fundamental open string with two ends can end on D-branes, so

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<sup>2</sup>We use the same name for a divisor, its associated line bundle and the first Chern class of this line bundle.

that the possible gauge groups are of  $U, SO, Sp$  types. However if a 7-brane carries NSNS charge as well, there can be zero modes of a tensionless string junction with more than two endpoints [32, 33], so the gauge symmetry can be enhanced to exceptional group.

It is useful to understand this in terms of intersection theory. Identifying the gauge group on  $S_i$  by  $\text{ord}(f, g, \Delta)$ , we write

$$\begin{aligned} F &= -4K_{B'} = (\text{ord } f_i)S_i + F', \\ G &= -6K_{B'} = (\text{ord } g_i)S_i + G', \\ D &= -12K_{B'} = (\text{ord } \Delta_i)S_i + \sum_R s_{R_i} D'_{R_i}. \end{aligned} \tag{4}$$

This decomposition  $F, G, D$  respectively shows the information on *dominant* terms in  $f, g, \Delta$ . The last term in  $D$  is the consequence of Tate's algorithm [8, 41]. Each matter  $R_i$  is localized on  $S_i \cdot D'_{R_i}$ , which is directly interpreted as the multiplicity of  $R_i$  in six dimensions. In geometric engineering, a gauge group is not only determined by a singularity type but also a monodromy condition [20]. The redundancy factors  $s_{R_i}$ , which we call splitness, contain the information, some of which are also displayed in Table 2. Later we will reconstruct  $s_{R_i}$  later (See (45) and below). Likewise, in the first two lines, the remaining part  $F'$  and  $G'$  contain the information on the leading order terms in  $f$  and  $g$ :  $S_i \cdot F'$  or  $S_i \cdot G'$  respectively give the order of the polynomials of the dominant terms in  $f$  and  $g$ . If some polynomial is a complete square or complete with some power, we say the singularity is split. Besides the gauge part, the entire rest part of the canonical class, including that of  $E'_8$  takes part in the formation of the matter curves.

This decomposition shows only the information on dominant terms. Knowing higher order terms means we have some information on an enhanced group which embraces the current one. It follows that, for a given group, vanishing of some parameter implies a symmetry enhancement. We parameterize, for instance, one higher order than (4) in  $f$  by the decomposition

$$F = (\text{ord } f_i + 1)S_i + (F' - S_i), \tag{5}$$

where we define a new primed quantity for the new expansion as the last term in the bracket. From the definition (2),

$$\text{ord } \Delta \geq \min(3 \text{ord } f, 2 \text{ord } g). \tag{6}$$

So, we can know which of  $f$  or  $g$  is dominant in  $\Delta$ , hence know whether  $\Delta$  is to be increased. This explains the inequality in Table 1. The equality in (6) does not hold when the leading order in  $\Delta$  is cancelled by those combinations of  $f$  and  $g$ . For example, for  $SO(10)$  the enhanced group can be  $SO(11)$  or  $E_6$ .<sup>3</sup> These correspond to the gauge symmetry enhancement directions, when we send a certain term in  $f$  or  $g$  to be zero. In

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<sup>3</sup>The  $SO(11)$  group is obtained from generic  $D_6$  singularity without splitness [8, 20].

such cases, information on an enhanced group is crucial. For example, in  $SO(10)$ , some element of  $\sum s_R D'_R$  comes from a next dominant term  $D'_R$  of  $E_6$ , indicating that the matter  $R$  of  $SO(10)$  is inherited from the branching of  $R'$  of  $E_6$  (See Subsec. 4.4). In the next subsection, we will follow the way of extensions.

In this work, we will consider a vanishing parameter that is *not necessarily the leading* term, enhancing the symmetry in a nontrivial way. There are also cases where the discriminant can be factorized into two or more factors, each representing a simple group or an Abelian group. So far, we have been interested in one factor, or one simple group, which is enough for the local unification group. We will make use of the fact that  $D$  can show not only the dominant contribution but also the *full components*, as the form (3) suggests. We will see that the global factorization is easily caught by divisor expansion and we can symmetrically describes all the factor group, including the subgroups of the other  $E_8$ . It is not impossible to read off the entire factors using the Tate's algorithm, which is mainly interested in one factor [8, 41], if we go back to some step and re-parameterize with respect to the position of another factor, taking care of higher order terms in the discriminant.

## 2.2 Weierstrass embedding and $E_8 \times E_8$

For elliptic Calabi–Yau manifold  $X_d$ , usually we make further *assumption* in the context of the duality to heterotic string:  $X_d$  allows a K3 fibration over a base  $B_{d-2}$  which is compatible with the elliptic fibration. Then, from the adiabatic argument, it is fiberwise dual to the heterotic string on a Calabi–Yau threefold, which is another elliptic fibration over the same base  $B_{d-2}$  [1, 35]. Since this K3 fiber is elliptically fibered over  $\mathbb{P}^1$ , this requirement narrows the possible base  $B'_{d-1}$  of elliptic fibration:

$$B'_{d-1} \text{ is a } \mathbb{P}^1 \text{ fibration over } B_{d-2}. \quad (7)$$

We will see that this requirement is quite strong, so that some features, like the gauge symmetry breaking pattern, is less sensitive to the choice of  $B_{d-2}$ . In fact such duality essentially points that the unification group is that of the dual heterotic string, namely  $E_8 \times E_8$  or  $SO(32)$ , up to possible enhancement from small instantons and dimensional reduction. Relaxing this assumption to obtain authentic F-theory vacua is another important question, which we shall not pursue here.<sup>4</sup>

Mostly we will deal with compactification on Calabi–Yau threefold  $X_3$ ,  $\pi' : X_3 \rightarrow B'_2$  [2, 3, 6, 31]. Because of the requirement (7), the base manifold can be completely specified to be the Hirzebruch surface  $\mathbb{F}_n$ , or  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$ . It is generated by two effective

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<sup>4</sup>We have a limited number of directly accessible Calabi–Yau fourfolds [37]. Recently uplifting of some class of type II orientifold models to F-theory has been developed [38–40]. D-branes and orientifolds are identified by divisors carrying an appropriate symmetry, so that global consistency condition is derived.

divisors  $r$  and  $t$ , satisfying the relations  $r \cdot t = 1$ ,  $r^2 = n$  and  $t^2 = 0$ .<sup>5</sup> Another irreducible divisor with negative self-intersection  $r_0 = r - nt$  is disjoint from  $r$

$$r \cdot r_0 = 0. \quad (8)$$

The canonical class is

$$-K_{B'_2} = r + r_0 + 2t. \quad (9)$$

We take  $z'$  and  $z$  as affine coordinates of the base  $\mathbb{P}^1$  and the fiber  $\mathbb{P}^1$ , respectively, such that  $t = \{z' = 0\}$  and  $r = \{z = 0\}$ . Near the base  $\mathbb{P}^1 = r$ , we have  $-K_{B'_2}|_r = -K_{B'_2} \cdot r = 2 + n$ . Since (8) implies  $r|_r = n$  and  $(r + nt)|_r = 0$ , from the transformation of  $f$  and  $g$ , we have

$$y^2 = x^3 + x \sum_{j=-4}^4 f_{4c_1-nj}(z') z^{4-j} + \sum_{k=-6}^6 g_{6c_1-nk}(z') z^{6-k}, \quad (10)$$

where  $c_1 \equiv c_1(B_1 = \mathbb{P}^1) = 2$ . The coefficients  $f_{8-nj}(z')$  and  $g_{12-nk}(z')$  are polynomials in  $z'$  of degree denoted by the subscripts, which we require to be non-negative, otherwise we understand such terms do not exist. Immediately we see, it is nothing but the deformation of  $E_8 \times E_8$  singularity

$$y^2 = x^3 + f_8(z') z^4 x + g_{12+n}(z') z^5 + g_{12}(z') z^6 + g_{12-n}(z') z^7. \quad (11)$$

In (10), two pairs  $z^{4\pm k}x$  and  $z^{6\pm k}$  are respectively exchanged to each other with opposite sign, by  $z \rightarrow 1/z$  up to rescaling  $x \rightarrow z^4x$  and  $y \rightarrow z^6y$ . This indicates this affine form comes from that in the projective space, where we can see the global structure [3]. Thus  $E_8 \times E_8$  is the maximal symmetry we can see from the Weierstrass equation in  $z$ .  $f_8$  and  $g_{12}$  are related to Kähler deformation on the heterotic side, and we take both of them very large while fixing  $f_8^3/g_{12}^2$  finite [3]. A quick check is that the possible deformation has the dimension  $8 + 12$  which is same as  $h^{1,1}$  of K3 in the heterotic dual.

In terms of divisors, two independent  $E_8$  symmetries, or  $\text{II}^*$  fibers, are supported by two disjoint curves  $r$  and  $r_0$ , thanks to the relation (8). From Table 1, each  $\text{II}^*$  singularity carries  $\text{ord}(f_i, g_i, \Delta_i) = (4, 5, 10)$ , so as many  $S_i$  are contained in  $F, G, D$ . Then we have the following partitions

$$\begin{aligned} F &= -4K_{B'_2} = 4r + 4r_0 + \underbrace{8t}_{F''}, \\ G &= -6K_{B'_2} = 5r + 5r_0 + \underbrace{r + r_0 + 12t}_{G''}, \\ D &= -12K_{B'_2} = 10 \underbrace{r}_{E_8} + 10 \underbrace{r_0}_{E'_8} + \underbrace{2r + 2r_0 + 24t}_{2D''_{\text{inst}}}. \end{aligned} \quad (12)$$

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<sup>5</sup>Conventional notations: sometimes  $\{C_0, C_\infty, f\}$  or  $\{D_v, D_u, f\}$  for  $\{r_0, r, t\}$  here.

The last line shows the component of discriminant locus (3). In view of  $E_8$  supported by  $r$ , as in (4), the relations  $r \cdot (4r_0 + F'') = 8$  and  $r \cdot (5r_0 + G'') = 12 + n$  shows the dominant terms in  $f$  and  $g$  respectively are  $f_8 z^4 x$  and  $g_{12+n} z^5$ . We have similar relations for the other  $E'_8$ , agreeing to (11). From the discriminant locus, we see

$$r \cdot (2r + 2r_0 + 24t) = 2(12 + n), \quad r_0 \cdot (2r + 2r_0 + 24t) = 2(12 - n), \quad (13)$$

showing  $g_{12+n}^2 z^{10}$  and  $g_{12-n}^2 z^{14}$  make up the discriminant. Thus we identify the ‘instanton curve’ as

$$r + r_0 + 12t. \quad (14)$$

This is a divisor interacting with two different  $E_8$ s, which is not easy to describe in the equation form. This connection leads to the well-known consistency condition

$$n_1 + n_2 + n' = 24 \iff \text{ch}_2(\mathcal{V}_1) + \text{ch}_2(\mathcal{V}_2) + n' = c_2(K3), \quad (15)$$

where  $n'$  is the number of possible blowups.

At the zeros of  $g_{12\pm n}(z')$ , the  $E_8$  singularity get worsen. This is interpreted as the effect of small instantons in the heterotic side [3, 42]. To describe such broken symmetry, we need either additional lower order terms of deformation, or blowing-up at that point. For the former, being lower order terms, they do not affect the instanton terms not modifying the relation (15) staying at  $n' = 0$ .

We can blow-up in the base to have zero size instantons. Blowing-up changes the intersection number  $n_1 = 12 + n$  and  $n_2 = 12 - n$  to different values, preserving the relation [24]. In the low-energy limit, the latter comprise the number of the tensor multiplets, not counting the one containing the dilaton [43]. The relation (15) corresponds to Bianchi identity in the heterotic side, where the instanton number is accounted by the second Chern class of a vector bundle  $\mathcal{V}_i$ . It guarantees the absence of anomalies. Each number of instantons  $n_i$  and tensor multiplets  $n'$  from the blowup to be positive, so that each cannot exceed 24. It is because the meaningful blowup is done at the intersections between  $E_8$  and the instanton curves. For  $n'$ , the original  $\mathbb{F}_n$  has no exceptional divisor of self-intersection  $(-1)$ , by Castelnuovo–Enriques Criterion [44], we cannot blow-down the primitive  $\mathbb{F}_n$ . The negative instanton number can be realized if we use analogous system as D7 and *anti*-D3 branes, but we have to face stability issue. The only possible source of larger symmetry than  $E_8$  is using an exceptional divisors, from the blow-up in the base, as components of the discriminant locus, e.g. in Ref. [20, 24].

If we have two global sections instead one, we have  $Spin(32)/\mathbb{Z}_2 \simeq SO(32)$  gauge symmetry. In this case, zero size instantons give rise to gauge symmetry enhancement over the original group [20, 42]. To obtain its subgroup is an interesting topic.



## 2.3 Four dimensions

From the constraint of heterotic dual (7), the Calabi–Yau fourfold has a similar structure as the above threefold [7]: This generalizes the construction of the Hirzebruch surface. To describe the geometry globally, we use the language of projective bundles. The base  $B'_3$  of the elliptic fibration is the total space of the projective bundle  $\mathbb{P}(\mathcal{O}_{B_2} \oplus t)$ , where  $t$  is a line bundle over  $B_2$ .<sup>6</sup> Thus  $B'_3$  is a  $\mathbb{P}^1$  fibration over  $B_2$ ,  $\pi'' : B'_3 \rightarrow B_2$ . Take  $r = \mathcal{O}(1)$  as the line bundle coming from the fiber  $\mathbb{P}^1$ . The sections describing  $r + t$  and  $t$  have no common zeros, so that

$$r \cdot (r + t) = 0. \quad (16)$$

Of course, we can express the discriminant locus in the same way. The total Chern class is, omitting pullback,

$$c(B'_3) = c(B_2)(1 + r)(1 + r + t) \quad (17)$$

meaning that

$$c_1(B'_3) = c_1(B_2) + 2r + t, \quad (18)$$

$$c_2(B'_3) = c_2(B_2) + c_1(B_2)(2r + t). \quad (19)$$

Thus  $D = 12c_1(B'_3) = 12c_1(B_2) + 24r + 12t$ . This is analogous condition as (8). This shows that, if we take the base coordinate as  $z$  such that  $r = \{z = 0\}$ , essentially the form of Weierstrass equation, as a projective form, is the same as in six dimensional case, with small replacement  $f_{4 \cdot 2 + nk} \rightarrow f_{4c_1(B_2) - kt}$  and  $g_{6 \cdot 2 + nj} \rightarrow g_{6c_1(B_2) - jt}$ . Then the Weierstrass equation essentially has the same form as (10) and the maximal gauge symmetry is  $E_8 \times E_8$ .

Knowing the details of the divisors supporting 7-branes, now surfaces in the base  $B'_3$  of Calabi–Yau fourfold, amounts to knowing the form of  $t$ . Let us call the set of generators of the divisors of  $B'_3$ , as  $\{s_i\}$ . We can replace  $t$  with the linear combinations of  $s_i$ 's

$$t = \sum_{s_i \cdot r \neq 0} a_i s_i, \quad a_i > 0. \quad (20)$$

The coefficients  $a_i$  are inherited from the divisor relations among  $\pi''(r), \pi''(s_i)$  in  $B_2$  [52]. For example if we do a  $\mathbb{P}^1$  fibration over  $\mathbb{F}_n$ , we have two more parameters  $k, m$  and  $t = k\sigma + mf$  for the generators of the divisors of  $\mathbb{F}_n$ . So a general divisor has a form  $S_i = r + a_\sigma\sigma + a_f f$  with some constants  $a_\sigma, a_f$ . Thus  $S_i$  have form  $S_i = r + \sum k_i s_i$  with  $|k_i| < a_i$ . We allow  $k_i$  for negative integer, interpreting as the intersection in the opposite orientation as in six dimensional case.

In four dimensional compactification, the intersection of 7-branes are surface in the base of elliptic fibration  $B'_3$ . To have four dimensional fermions, we need more input giving

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<sup>6</sup>Again we use the same name for the line bundle, its first Chern class and the associated divisor.

rise to four dimensional chirality, for example from the magnetic flux background [11, 12]. Also we have an extra condition on anomaly cancellation on intersections having 3-brane charges, which is in general singular [45–47]. Some components suffer worse singularity, whose remedy is recently discussed in [38, 39].

### 3 Matter curves

Once the decomposition of the discriminant locus (3) is determined, like arranging 7-branes, we should be able to obtain the matter content. From the mechanism of Katz and Vafa [9], matter fields are localized along the intersections of two 7-branes. For two gauge groups whose support are given by divisors  $S_i$  and  $S_j$ , we have the expansion

$$S_i \cdot S_j = \sum_{R, R'} w_{R_i, R'_j} \Sigma_{R_i, R'_j}, \quad (21)$$

where  $\Sigma_{R_i, R'_j}$  are the effective intersections of the bases spanning  $S_i$  and  $S_j$ . The problem is, RHS of (21) requires more information, that is,  $w_{R, R'}$ , while LHS gives at best  $\Sigma_{R, R'}$ . In the next subsection, we will learn that the relative ratio of  $w_{R, R'}$  is determined by Green–Schwarz mechanism. Then, in Subsec. 3.2, we consider this problem considering enhanced gauge groups.

#### 3.1 Green–Schwarz mechanism

The first guideline is the anomaly cancellation structure of Green–Schwarz (GS) mechanism in six dimensions [22]

$$\ell(\text{adj}_i) - \sum_R \ell(R_i) n_{R_i} = 6K_{B'} \cdot S_i, \quad (22)$$

$$y_{\text{adj}_i} - \sum_R y_{R_i} n_{R_i} = -3S_i^2, \quad (23)$$

$$x_{\text{adj}_i} - \sum_R x_{R_i} n_{R_i} = 0 \quad (24)$$

$$\sum_{R, R'} \ell(R_i) \ell(R'_j) n_{R_i R'_j} = S_i \cdot S_j. \quad (25)$$

Here  $\ell(R)$  is the index of the representation  $R$ ,  $\text{tr}_R t_a t_b = \ell(R) \delta_{ab}$ , and the quartic and the quadratic invariants define the coefficients  $\text{tr}_R F^4 = x_R \text{tr} F^4 + y_R (\text{tr} F^2)^2$ . By trace without subscript we mean the minimal representation, i.e. the fundamental representation of  $SU(n)$ . We use absolute normalization reflecting the branching under the subgroup  $R = \bigoplus(r, r') \implies \ell(R) = \sum_r \ell(r) \dim r'$  by defining property of trace. Thus, for the breaking  $E_6 \rightarrow SO(10)$ , exactly  $\ell(\mathbf{27}) = \ell(\mathbf{16}) + \ell(\mathbf{10}) + \ell(\mathbf{1})$  holds, for instance. Similar relations

hold for  $x_R$  and  $y_R$ . Thus the condition (24) gives us well-known irreducible gauge anomaly cancellation. In Table 2, we displayed relevant invariants for some typical gauge group and representations.

In six dimensions, a matter curve is a point, so in our notation (25) defines the expansion. We see that in the decomposition of the matter curve the group factors are included. For instance, **27** of  $E_6$  has  $\ell(\mathbf{27}) = 6$ , so in the parallel separation  $(\mathbf{2}, \mathbf{27})$  contributes  $6n_{(\mathbf{2}, \mathbf{27})}$  in the decomposition. It is a generalization of the pair of mirror branes on top of orientifold planes if we have vector representation of  $SO(n)$  type gauge symmetry. Eq. (22) shows the matter content of a given brane is constrained by the product of the cycle and the canonical class. These are already strong constraints to limit possible models. In the simplest base without genus we have no higher order representation than fundamental for  $SU(n)$  case. On the heterotic side, in the level one Kac–Moody algebra, we can have spinorial and vector in  $SO(2n)$  representation, and antisymmetric tensor representations of  $SU(n)$  [48].

The GS conditions also give us information on what kind of divisor can support the gauge group. They are derived [22] from that a number of antisymmetric tensor fields provide the Poincaré dual basis to the first Chern class, entered in LHS of (3) and gauge groups are supported by the number of 7-branes, in RHS. For  $\mathbb{F}_n$  we have two two-cycles  $h^{1,1} = 2$ , whose orthonormal basis is provided by  $r/\sqrt{n}$  and  $r_0/\sqrt{n}$ . We can decompose

$$\begin{aligned} c_1(B_2) &= \frac{n-2}{n}r + \frac{n+2}{n}r_0, \\ S_i &= a_i \frac{r}{\sqrt{n}} + b_i \frac{r_0}{\sqrt{n}}. \end{aligned} \tag{26}$$

We can make 4-form which enters a factor in the anomaly polynomial in  $I_8$ ,  $(n-2)\text{tr}R^2/2\sqrt{n} + 2a_i\text{tr}F_i^2, (n+2)\text{tr}R^2/2\sqrt{n} + 2b_i\text{tr}F_i^2$ . Then we can cancel the anomaly by counterterm including antisymmetric tensor fields  $H^1 = dB^1 + (-n+2)\omega_{3L}/2\sqrt{n} + 2a_i\omega_{3Y}^i, H^2 = dB^2 + (-n-2)\omega_{3L}/2\sqrt{n} + 2b_i\omega_{3Y}^i$ , where  $\omega_{3L}, \omega_{3Y}$  are respectively a gravitational and a Yang–Mills Chern–Simons form. We note that, at least formally, there is no requirement for  $a_i, b_i$  to be positive integers or rational number.

### 3.2 Heterotic calculation and inheritance condition

If a gauge group obtained from F-theory is semisimple, in general, it is not sufficient to determine the multiplicity of a matter charged under more than one simple group. It is because the relation (25) gives only the total number of such matters for given two groups. We can learn how to remedy it by studying the spectrum on the corresponding heterotic dual model. The condition (7) implies that there exists a heterotic dual for our construction on the Hirzebruch surface. On the heterotic side, a semisimple group occurs

if we have a line bundle background. It has its own importance, since many realistic models are obtained in this context.

First, we specify the embedding of the line bundle  $\mathcal{L}$ . The ‘Wilson line’ vector  $V$  contains the information on the embedding in the Cartan directions<sup>7</sup>

$$V = (V_1, V_2, \dots, V_8) \iff \mathcal{V} = (\mathcal{L}^{V_1}, \mathcal{L}^{V_2}, \dots, \mathcal{L}^{V_8}) \quad (27)$$

for each  $E_8$ . Contrary to F-theory case where the gauge group is arbitrary, here it is completely specified by breaking of  $E_8 \times E_8$ . We require  $\text{ch}_2(\mathcal{V}) = \frac{1}{2}V^2 c_1(\mathcal{L})^2 = 12 \pm n$  for anomaly cancellation (15). This data specifies the spectral cover for  $E_8$  [6, 7, 12]. For the standard weight vectors  $w$  of  $E_8$  [48], the relation  $V \cdot w = 0$  determines the gauge bosons of unbroken group.

The matter spectrum is unambiguously computed by various bundle cohomology groups. In our case, the only necessary information is the charge  $q = V \cdot w$  of the structure group. From the branching  $\mathbf{248} \rightarrow \bigoplus r_q$ , the number of multiplet is obtained by index theorem,

$$\begin{aligned} n_q &= \int_{\text{K3}} \text{Todd}(\mathcal{M}) \text{ch}(\mathcal{V}) \\ &= \int \left( \frac{q^2}{V^2} \text{ch}_2(\mathcal{V}) - \frac{1}{12} c_2(\mathcal{M}) \right) \\ &= \int \frac{q^2}{2} c_1(\mathcal{L})^2 - \frac{1}{12} \chi(\text{K3}) \\ &= \frac{q^2}{V^2} (12 \pm n) - 2. \end{aligned} \quad (28)$$

Physics is independent of the normalization, because only the relative size  $V/q$  matters which does not change under Weyl reflections. We also used the Calabi–Yau condition  $c_1(\text{K3}) = 0$ , and  $c_2(\text{K3}) = \chi(\text{K3}) = 24$  is the Euler number of K3. For the nonabelian gauge bundle  $\mathcal{V}$ , we do not need the prefactor  $q^2/V^2$  and  $\text{ch}_2(\mathcal{V}) = c_2(\mathcal{V})$ .

If there are more than two matter fields from the branching, there is insufficient information for the relative relations on  $n_{R,R'}$ . This happens because we have a *semisimple* commutant to a  $U(1)$  background. If we have a simple unbroken gauge group, there is always a unique branching for a fixed  $U(1)$  charge. So we will use a principle, ‘*equal inheritance condition*,’ that the relative relation comes from the unified group. Namely, if a given group is a common proper subgroup of two enhanced groups  $G_{M1} \cap G_{M2} = G$ , a representation  $R \in G$  is inherited by branching of  $R_{M1} \in G_{M1}$  and  $R_{M2} \in G_{M2}$ . Then, the multiplicity is equally inherited from the original representations,

$$n_R = \frac{1}{2} n_{R_{M1}} + \frac{1}{2} n_{R_{M2}}. \quad (29)$$

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<sup>7</sup>For a similar construction in different context, see, e.g. [15].

For example,  $(\mathbf{2}, \mathbf{16})$  of  $U(2) \times SO(10)$  comes from the branching of  $(\mathbf{2}, \mathbf{27})$  of  $U(2) \times E_6$ , and also from of  $(\mathbf{3}, \mathbf{16})$  of  $SO(10) \times U(3)$ . So we have the multiplicity  $n_{(\mathbf{2}, \mathbf{16})} = \frac{1}{2}n_{(\mathbf{2}, \mathbf{27})} + \frac{1}{2}n_{(\mathbf{3}, \mathbf{16})}$ . This is justified by the index theorem

$$n_{q_1, q_2} = \int \left( \frac{1}{2} \frac{q_1^2}{V_1^2} + \frac{1}{2} \frac{q_2^2}{V_2^2} \right) (12 + n) - 2. \quad (30)$$

This happens only under the line bundle background, and the spectrum is completely determined solely by the  $U(1)$  charges. This redefines the  $U(1)$  normalization, under the condition that the ‘instanton’ number for the line bundle  $(12 + n)$  should not be changed, and the second term on the RHS should always be the same for the line bundle  $-c_2(K3)/12$ . In the limit  $V_1 = V_2$ ,  $q_1 = q_2$  it should reduce to the original form (28). An applied example is given at the end of Subsec. 4.4.

Finally, the absence of six dimensional gravitational anomaly imposes the condition on the number of tensor  $n_T$ , hyper  $n_H$  and vector multiplets  $n_V$ ,

$$29n_T + n_H - n_V = 273. \quad (31)$$

It is automatically satisfied if the gauge symmetry is obtained from breaking of  $E_8$ , satisfying the constraint, LHS of (15). The dimension of the moduli space of  $E_8$  with the instanton number  $n_1 = 12 + n$  is  $\dim \mathcal{M}_{E_8}(n_1) = 30n_1 - 248 = 30n + 112$ . It does not change by spontaneous symmetry breaking, so that  $n_H - n_V$  is preserved. A blow-up increases the number of tensor multiplet so that  $n_T = h^{1,1} - 1$ , and  $n_T = n' + 1$  for  $\mathbb{F}_n$ . With 20 deformations from  $f_8$  and  $g_{12}$ , we have the relation  $n_H - n_V = 20 + (n_T - 1) + \dim \mathcal{M}_{E_8}(n_1) + \dim \mathcal{M}_{E_8}(n_2) = 273 - 29n_T$  automatically satisfying the constraint (31).

### 3.3 $u(1)$ brane

We always have to face a  $U(1)$  symmetry if we consider a semisimple group. The easiest way to understand this is using the heterotic dual. A background line bundle yields unbroken group as the commutant in the unified group. This  $U(1)$  commutes to itself and survives [15, 16].

In the perturbative limit, the  $U(1)$  group is related to the overall center of momentum motion of a D-brane stack. Thus there is no independent degree of freedom for this. Since we obtain such symmetry if we make a subleading parameter to be zero, we may also employ a virtual 7-brane  $Q_j$  responsible for the  $U(1)$  symmetry as the corresponding divisor for the parameter set to be zero. We have a modified relation corresponding to (25)

$$\sum_{R, q} \ell(R_i) \frac{q_j^2}{V^2} n_{R_i q_j} = S_i \cdot Q_j. \quad (32)$$

It follows that the very existence of such brane predicts that the RHS should be universal for every other group.

### 3.4 Brane reduction

From the eight dimensional twisted supersymmetry, we have the field  $\phi_{mn}$  transforming as  $K_{B'} \otimes \text{adj} P$  on  $B'$  [11]. Its nontrivial profile  $\langle \phi_{mn} \rangle$  is naturally interpreted as the deformation of the 7-branes in the normal direction. We do not need to know its field theoretic description in detail, since we have explicit description of gauge group in terms of Weierstrass equation. The point is that  $\phi_{mn}$  transforms as the adjoint representation. So we expect at least for simplest profile we have gauge symmetry breaking of  $E_8$  without rank reduction. We have analogy for the D-brane stack in the flat space, where the constant VEVs of normal scalar to the D-brane correspond to parallel separation of the branes. Higgsing it leads us to the symmetry breaking and enhancement  $G \leftrightarrow H_1 \times H_2$ , the adjoint branches as

$$\mathbf{adj} G \leftrightarrow (\mathbf{adj} H_1, \mathbf{1}) + (\mathbf{1}, \mathbf{adj} H_2) + (\text{off-diagonal comp.}) + \text{c.c.} \quad (33)$$

The matter representations come from the branching of the adjoint. The well-known example is  $U(m+n) \rightarrow U(m) \times U(n)$ .

$$(\mathbf{m} + \mathbf{n})^2 - \mathbf{1} \rightarrow (\mathbf{m}^2 - \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{n}^2 - \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{m}, \mathbf{n}) \oplus (\overline{\mathbf{m}}, \overline{\mathbf{n}}).$$

The off-diagonal elements  $(\mathbf{m}, \mathbf{n})$  or  $(\overline{\mathbf{m}}, \overline{\mathbf{n}})$  are in general not vectorlike because each belong to different cohomology with different dimension. It depends on how the transition takes place. If the symmetry breaking is spontaneous, we can think of the reverse process as symmetry enhancement. The general form is

$$y^2 = x^2 + \prod_{l=1}^m (z - u_l) \prod_{k=1}^n (z - t_k). \quad (34)$$

This is viewed as a local form of the Weierstrass equation. If all  $u_l$  and  $t_k$  assume the same value, say 0, the symmetry is enhanced to  $A_{m+n-1}$

$$y^2 = x^2 + z^{m+n}, \quad (35)$$

i.e.,  $\text{ord}(f, g, \Delta) = (0, 0, m+n)$ . Thus  $u_l$  and  $t_k$  in (34) parameterize deformations of (35). Among deformations, if all  $u_l$  have the same value 0, and all  $t_k$  have another same value  $t \neq 0$ , the deformed curve describes  $A_{m-1} \oplus A_{n-1}$  symmetry

$$y^2 = x^2 + z^m (z - t)^n. \quad (36)$$

Around  $z = 0$ ,  $(z - t)$  is fixed to be nonzero, thus the curve looks as that of  $A_{m-1}$ . The same holds for  $A_{n-1}$  around  $z = t$ . In particular if  $t$  is a complex number, the surfaces  $\{z = 0\}$  and  $\{z = t\}$  is homologous. We can generalize it to the case of exceptional groups [23, 49]. In terms of divisors, we have

$$(m+n)S \longrightarrow mS_1 + nS_2 \quad (37)$$

The divisors  $S, S_1, S_2$  may not be same or linearly equivalent.

There are some components which are not responsible for the gauge dynamics. The deformation due to finite size instanton is rather close to brane recombination, a la [11], triggered by the field on the defect developing VEV. In D-brane case,  $k$  instantons embedded in  $U(N)$  is described by  $k$  D0 on  $N$  D4 branes (or its T-duals) [17, 51]. If we just place D0s, we just have zero size instantons and there is gauge symmetry enhancement to  $U(k) \times U(N)$ . If we grow the size of instantons, it is translated into assigning nonzero VEVs to  $(\mathbf{k}, \mathbf{N})$  representations. The size-growing of instanton is clearer if we take T-dual in some two direction, then the initial system looks like two stacks of intersecting branes and the bifundamental fermion  $(\mathbf{k}, \mathbf{N})$  is one localized at the angles. The resulting gauge symmetry is broken down to  $U(k-l) \times U(N-l)$  undergoing the rank reduction. We describe this case

$$kS_1 + NS_2 \longrightarrow (k-l)S_1 + (N-l)S_2 + l(S_1 + S_2)$$

where the last term does not support the gauge degree of freedom. The corresponding Weierstrass equation is obtained

$$y^2 = x^2 + z^k z'^N \rightarrow y^2 = x^2 + z^{k-l} z'^{N-l} \prod_{i=1}^l (zz' + b_i). \quad (38)$$

Here we set deformation parameter with poles of order  $l$  [11], because coordinate dependent deformation on the off-diagonal components can be diagonalized with poles. Needless to say,  $b_i \rightarrow 0$  recovers the original symmetry. This is not trivial if we embed this curve in  $E_8$ , where we will see that we have a very large parameter forcing  $b_i \rightarrow 0$ . See Subsec. 4.2.

The splitness  $s_R$  contains information on, to which group the given group is embedded. If the gauge group at hand is embedded in exceptional group, it is far from trivial since the notion of parallel is not trivial in general. Consider the above transition

$$D = kS_1 + NS_2 + D''. \quad (39)$$

Since  $kS_1 + D''$  and  $NS_2 + D''$  are  $I_0$  singularities, we have  $n_{\mathbf{k}} = S_1 \cdot (NS_2 + D'')/s_{\mathbf{k}}$  and  $n_{\mathbf{N}} = S_2 \cdot (kS_1 + D'')/s_{\mathbf{N}}$ . In the next section (and also in Ref [8]) we see  $s_{\mathbf{k}}$  is not always 1. Since we know  $n_{\mathbf{k}, \mathbf{N}} = S_1 \cdot S_2$ , we have

$$\begin{aligned} s_{\mathbf{k}} n_{(\mathbf{k}, \mathbf{1})} &= N(s_{\mathbf{k}} - 1)S_1 \cdot S_2 + S_1 \cdot D'', \\ s_{\mathbf{N}} n_{(\mathbf{1}, \mathbf{N})} &= k(s_{\mathbf{N}} - 1)S_1 \cdot S_2 + S_1 \cdot D''. \end{aligned} \quad (40)$$

The recombination is done by giving VEVs to  $l$  multiplets of  $(k, N)$ . We have the branching  $(\mathbf{k}, \mathbf{1}) \rightarrow (\mathbf{k} - \mathbf{1}, \mathbf{1}) + l(\mathbf{1}, \mathbf{1})$ ,  $(\mathbf{k}, \mathbf{N}) \rightarrow (\mathbf{k} - \mathbf{1}, \mathbf{N} - \mathbf{1}) + l(\mathbf{1}, \mathbf{N} - \mathbf{1}) + l(\mathbf{k} - \mathbf{1}, \mathbf{1}) + l^2(\mathbf{1}, \mathbf{1})$ . Thus we know

$$n_{\mathbf{k}-\mathbf{1}} = n_{(\mathbf{k}, \mathbf{1})} + N n_{(\mathbf{k}, \mathbf{N})} \quad (41)$$

and a similar for  $n_{\mathbf{N}-1}$ . The divisor (39) undergoes the reduction to

$$D = (k-l)S_1 + (N-l)S_2 + \underbrace{D'' + l(S_1 + S_2)}_{D''_{\text{new}}}, \quad (42)$$

where  $D''_{\text{new}}$  makes up a new divisor. Thus, we have

$$s_{\mathbf{k}-1}n_{\mathbf{k}-1} = S_1 \cdot ((N-l)S_2 + D''_{\text{new}}) = NS_1 \cdot S_2 + S_1 \cdot D'' + lS_1^2. \quad (43)$$

Equating it with (41), we have

$$lS_1^2 = N(2 - s_{\mathbf{k}-1} - s_{\mathbf{k}})n_{(\mathbf{k}, \mathbf{N})} + (s_{\mathbf{k}} - s_{\mathbf{k}-1})n_{(\mathbf{k}, \mathbf{1})}.$$

If we assume  $s_{\mathbf{k}} = s_{\mathbf{k}-1} = 1$ , which is the case in the perturbative  $SU(n)$  type group, immediately we see the self-intersection of  $S_1$  should be zero. This is the case when we embed D-branes on flat torus where parallel branes have no intersection. It also means that we cannot embed these  $U(k)$  groups on the divisor  $r$  of Hirzebruch space  $\mathbb{F}_n$  with  $n \neq 0$ .

We may generalize such mechanism for exceptional group. For the reduction of the divisor, the orders of polynomials  $\text{ord}(f, g, \Delta)$  are preserved. If the Dynkin diagram allows, there is always corresponding symmetry breaking [50]. The remaining information is monodromy or splitness conditions. To see collision between different gauge groups, we further decompose the discriminant locus.

$$D = -12K_{B'} = \sum_i (\text{ord } \Delta_i) S_i + \sum_{k, U(1)} Q_k + \dots, \quad (44)$$

where, inevitably we always have a 7-brane  $Q_k$  parameterizing  $U(1)$  dynamics. Still, in the description employing the divisors and intersections, we can symmetrically discuss the divisors supporting factor groups.

We have two independent ways of obtaining the matter spectrum. In particular if there is only one kind of matter representation  $R$ , the product  $n_R = S_i \cdot D''/s_i = (\ell(\text{adj}_i) - 6K_{B'} \cdot S_i)/\ell(R)$ . In this way we can inversely reconstruct Tate's algorithm, since knowing the redundancy  $s_R$  for each matter  $R$  completely specifies the required splitness. Plugging the GS conditions (22)-(25) into (4),

$$6\ell(\text{adj}_i) - \text{ord } \Delta_i y_{\text{adj}_i} = \sum_R (6\ell(R_i) - \text{ord } \Delta_i y_{R_i} - 3s_{R_i}) n_{R_i}. \quad (45)$$

Since we know all the information from the group theory and the orders of singularity, we can obtain the splitness from this equation. If there are several branched representations we go to enhanced group to input more information.



## 4 Extended gauge symmetry

In this section, we first review the engineering of simple subgroups of  $E_8$  on the Hirzebruch surface [8], reinterpreted in terms of intersection theory [31]. From the discussion in the previous section, we can identify the gauge symmetry by the data (4). We can generalize the description to semisimple, Abelian and non-simply laced group.

### 4.1 $E_6$

The first example is  $E_6$  and we can similarly analyze  $E_7$ . We assume that the corresponding singularity is located along the divisor  $r$ . Later we will relate the mother group  $E_8$  in the previous section, sitting along the same divisor. From Table. 1,  $E_6$  has  $\text{ord}(f, g, \Delta) = (3, 4, 8)$ . Thus we may write down two partitions

$$\begin{aligned} F &= 3r + \underbrace{r + 4r_0 + 8t}_{F'}, \\ G &= 4r + \underbrace{2r + 6r_0 + 12t}_{G'}, \\ D &= 8 \underbrace{r}_{E_6} + \underbrace{4r + 12r_0 + 24t}_{4D'_{27}}. \end{aligned} \tag{46}$$

From  $r \cdot F' = n + 8$ , the dominant term in  $f$  is  $f_{8+n}(z')z^3x$ , and from  $r \cdot G' = 2n + 12$  we see the dominant term in  $g$  is  $g_{12+2n}(z')z^4$ , which is also the leading term in the discriminant. It is seen from  $r \cdot 4D'_{27} = 4n + 24 = r \cdot 2G' = 2(2n + 12)$ . The splitting condition tells us that the matter curve is further reduced to  $(2r + 12t)/2 = r + 6t$ , or  $g_{12+2n} = q_{6+n}^2$ . The resulting equation is

$$y^2 = x^3 + f_{8+n}z^3x + q_{6+n}^2z^4 + O(z^5). \tag{47}$$

The discriminant has the form

$$\Delta = 27z^8q_{6+n}^4(z') + O(z^9). \tag{48}$$

We have  $(6 + n)$  localized matters **27** along the zeros of  $q_{6+n}(z')$ . Heterotic string on K3 independently gives such the result, for which higher order terms in  $O(z^9)$  are irrelevant, i.e. if we embed an instanton in  $SU(3)$ , we have the same spectrum. However this is just *one* of the possible solutions, only when we assume that the higher order terms are *generic*, i.e. not factorized any more. We will consider more general case in the following.

We can blow-up at the intersection points. Like instanton, we do not need details of the position, but the embedding group and the total number of blow-up sufficiently specify the physics. Suppose we blow-up at  $n'$  points, then the relation changes the relations  $\hat{r}^2 = n - n'$ ,  $\hat{t}^2 = -n'$ ,  $-\hat{K} \cdot \hat{r} = -n - 2 + n'$  while leaves  $\hat{r} \cdot \hat{t} = 1$ . Thus the number of **27** becomes  $\hat{r} \cdot (\hat{r} + 6\hat{t}) = n - n' + 6$ . We see that still there is no anomaly.

## 4.2 $E_6 \times U(2)$ and factorization

In (47), note that tending  $q_{6+n} \rightarrow 0$  enhances the symmetry to  $E_7$ ,

$$y^2 = x^3 + f_{8+n}z^3x + O(z^5).$$

If we had no  $f_{8+n}z^3x$  term in (47), the gauge symmetry would be  $U(2) \times E_6$ , because then  $q_{6+n} \rightarrow 0$  enhances the symmetry to  $E_8$ . We will see later we also need  $g_{12+n} = 0$ , clearly interpreted as zero instanton number in the heterotic side. Considering full factors up to  $O(z^6)$  in the equation, it is described by

$$y^2 = x^3 + f_8z^4x + q_{6+n}^2z^4 + g_{12}z^6 + O(z^7). \quad (49)$$

Keeping to  $O(z^{10})$  the discriminant becomes<sup>8</sup>

$$\Delta = 54z^8q_{6+n}^2g_{12}(z^2 + q_{6+n}^2/2g_{12}) + O(z^{11}). \quad (50)$$

The factor in the bracket shows the degree two  $A_1$  singularity factor  $z^2$  up to finely broken effect by  $q_{6+n}^2/g_{12}$ , since we are working in the limit  $g_{12} \rightarrow \infty$ . If we really remove  $q_{6+n}$  and restore  $g_{12+n}$  term, we again recover  $E_8$ . Later we will see evidences for surviving  $SU(2)$  with explicit examples. This unattractive form is due to the fact that we have deformation in  $y$ , see for example Ref. [9]. This is formally parameterized as  $\text{ord}(f, g) = (0, 0)$  for  $I_n$  or  $A_{n-1}$  singularity, and to see the full symmetry, we should refer to the equation in Tate's form, which we do not need for our discussion [8, 41].

We restate the above in terms of divisors. Since we have no  $f_{8+n}z^3x$  term, the corresponding divisor  $F$  has is expanded around  $4r$ , instead of  $3r$ . Still  $\text{ord } \Delta = 8$  as in (6). Putting  $2r$  for  $A_1$  singularity, the remaining part is  $2r + 24t + 2r_0$ . Since we have no terms except  $f_{6+n}$ ,  $f_8$  and  $g_{12}$ , only we can expand as

$$\begin{aligned} F &= 4r + 0 + 4r_0 + 8t, \\ G &= 4r + 0 + 5r_0 + 2(r + 6t) + r_0, \\ D &= 8 \underbrace{r}_{E_6} + 2 \underbrace{r}_{A_1} + 10 \underbrace{r_0}_{E'_8} + 2(r + 6t) + 2r_0 + 12t. \end{aligned} \quad (51)$$

We assume the other  $E'_8$  is unbroken for convenience, which is in fact unnecessary since all the divisors for  $E_6$  and  $SU(2)$  is orthogonal to  $r_0$  supporting  $E'_8$ . Therefore we can interpret this situation as the symmetry breaking of  $E_8$  via transition

$$10r \longrightarrow 8r + 2r. \quad (52)$$

Two  $r$ 's on the RHS respectively support  $E_6$  and  $SU(2)$ , which are not necessarily the same curves, as (50) shows; Better say, they are *different* curves that are *linearly equivalent*. On

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<sup>8</sup>In terms of [23], the deformation parameters satisfy the relation  $f_{E_8}(U; t) = f_{E_6} \cdot f_{A_1}(U - \tau_1/3 - \mu_1/9; t'')$ .

$\mathbb{F}_n$ , linearly equivalent curves can have nonzero net intersections. Their intersection number is  $n$ , so we have as many  $(\mathbf{2}, \mathbf{27})$  divided by the group theoretical factor  $\ell(\mathbf{2})\ell(\mathbf{27}) = 6$ , as in (25). Obviously, VEVs of two  $(\mathbf{2}, \mathbf{1})$ 's would completely break  $SU(2)$ , reducing to the previous  $E_6$  model having  $(n+6)$   $\mathbf{27}$ 's, *neglecting* the other factor group  $SU(2)$ . Thus we have net number of  $(\mathbf{1}, \mathbf{27})$  as  $(n+6) - \frac{n}{6}(\dim \mathbf{2})$ . Similarly we can calculate the number of  $(\mathbf{2}, \mathbf{1})$ 's  $16 + 6n - \frac{n}{6}(\dim \mathbf{27})$ , therefore

$$\frac{n}{6}(\mathbf{2}, \mathbf{27})_1 + \frac{1}{3}(18 + 2n)(\mathbf{1}, \mathbf{27})_{-2} + \frac{1}{2}(32 + 3n)(\mathbf{2}, \mathbf{1})_{-3}. \quad (53)$$

We verify that it is consistent with the GS relations (22)-(24). Sometimes fractional multiplicity indicates localization of the matter along a part of the geometry. In our calculation the divisor is globally given, so we can have meaningful spectrum if  $n$  is a multiple of 6. Only the positive matter multiplicity is allowed in six dimension, since there is only one possible chirality for the matter. When  $n = 0$  we have no localized  $(\mathbf{2}, \mathbf{27})$ , which is analogous to ‘parallel separation’ of D7-branes in the perturbative description without bifundamental zero mode.

We singled out  $u(1)$  divisor  $(r + 8t)$  in (51), provided by the missing  $f_{8+n}$  term in (32). We can track its origin from the enhanced group  $E_7 \rightarrow E_6 \times U(1)$ . As a consistency check, we calculate the GS conditions with respect to  $E_6 \times U(1)$  and  $SU(2) \times U(1)$  respectively giving

$$\begin{aligned} \frac{n}{6}(\dim \mathbf{2})\ell(\mathbf{27}) \cdot 1^2 + \frac{1}{3}(18 + 2n)(\dim \mathbf{1})\ell(\mathbf{27}) \cdot (-2)^2 &= 18(n + 8), \\ \frac{n}{6}(\dim \mathbf{27})\ell(\mathbf{2}) \cdot 1^2 + \frac{1}{2}(32 + 3n)(\dim \mathbf{1})\ell(\mathbf{2}) \cdot (-3)^2 &= 18(n + 8). \end{aligned} \quad (54)$$

From the difference of the number of charged hypermultiplets and vector multiplets, we have

$$\begin{aligned} [n_H - n_V]_{E_6 \times U(2)} &= \frac{n}{6} \cdot 2 \cdot 27 + \frac{1}{3}(18 + 2n) \cdot 27 + \frac{1}{2}(32 + 3n) \cdot 2 - 78 - 3 - 1 \\ &= 30n + 112, \end{aligned} \quad (55)$$

showing that these are the symmetry obtained from  $E_8$ . Since there is no singlet hypermultiplets, we have consumed all the moduli; We have to tune all the coefficients to have the unbroken group. This also indicates that  $E_6 \times U(2)$  is the maximal subgroup including  $E_6$ .<sup>9</sup> Note on the fixed moduli:  $q_{6+n}$  is completely tuned, leaving no massless field.

On the heterotic side, this model corresponds to one under the line bundle background. There is no zero in  $g_{12+n}(z') = 0$ , meaning that the second Chern class is  $c_2 = 0$  for the background bundle. In this context we have  $U(2) \times E_6$  as the maximal subgroup containing  $E_6$ . To provide the appropriate amount to the Bianchi identity (15), we interpret that the line bundle  $\mathcal{L}$  gives  $\text{ch}_2(\mathcal{V}) = V^2 c_2(\mathcal{L})^2 / 2 = 12 + n$ , with  $V^2 = (2, 1, 1, 0^5)^2 = 6$ . Every

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<sup>9</sup>According to Ref. [23], this is allowed as the maximal, for the purely algebraic reason.

subgroup here commutes to  $U(1)$  in  $E_8$  to which the line bundle is embedded. Using the index theorem (28), we can calculate the spectrum on the heterotic side

$$n_q = q^2(12 + n)/V^2 - 2,$$

thus

$$\begin{aligned} (\mathbf{2}, \mathbf{27})_1 : n_1 &= \frac{1}{6}(12 + n) - 2 = \frac{n}{6}, \\ (\mathbf{1}, \mathbf{27})_{-2} : n_{-2} &= \frac{(-2)^2}{6}(12 + n) - 2 = \frac{1}{3}(18 + 2n), \\ (\mathbf{2}, \mathbf{1})_{-3} : n_{-3} &= \frac{(-3)^2}{6}(12 + n) - 2 = \frac{1}{2}(32 + 3n), \end{aligned}$$

agreeing with (53). It is not a coincidence that the matter multiplicity and charge quantization is very similar. We can understand the line bundle plays a similar role as the instanton, giving the dimension of the moduli space  $30\text{ch}_2(\mathcal{V}) - 248 = 30n + 112$ .

This description shows that we can investigate the behavior of both singularities of  $E_6$  and  $A_1$  on the equal footing. The role of divisors carrying each of them is equal. In this example, two groups are supported by linearly equivalent divisors, so except the order of singularity the matter multiplicity was symmetric. There is a case where the subgroup of  $E_8$  is not supported along the original divisor for  $E_8$ .

### 4.3 Deformation of positions and matching the full $E_8 \times E_8$

It is not compulsory for  $E_6$  to lie along the divisor  $r$ . The GS condition showed that the divisor supporting a gauge field is just expanded by two-cycles with arbitrary coefficients. We may consider for example  $(r - 2t)$ ,

$$\begin{aligned} F &= 3(r - 2t) + \underbrace{r + 4r_0 + 14t}_{F'}, \\ G &= 4(r - 2t) + \underbrace{2r + 6r_0 + 20t}_{G'}, \\ D &= 8\underbrace{(r - 2t)}_{E_6} + \underbrace{4r + 12r_0 + 40t}_{4D'_{\mathbf{27}}}. \end{aligned} \tag{56}$$

From the intersection number  $(r - 2t) \cdot (4r + 12r_0 + 40t) = 4(n + 2)$  with the same splitness 4. Thus the number of  $\mathbf{27}$  is  $n + 2$ . The spectrum is consistent with GS conditions (22)-(25). We know that, from the  $E_7$  mother group, we have order  $n + 6$  polynomial for  $r + 8t$ . From the  $E_8$  mother group, we have instanton number  $(r - 2t) \cdot (r + r_0 + 12t) = 8 + n$ . We can verify the total dimension of the moduli space

$$(n + 2) \cdot 27 - 78 + (8 + n) + (4 + n) + (2 + n) = 30n - 8.$$

This dimension is also obtainable from a model with  $E_6 \times U(2)$  all localized at  $(r - 2t)$ . The spectrum is

$$\frac{1}{6}(-4 + n)(\mathbf{2}, \mathbf{27})_1 + \frac{1}{3}(10 + 2n)(\mathbf{1}, \mathbf{27})_{-2} + \frac{1}{2}(20 + 3n)(\mathbf{2}, \mathbf{1})_{-3}. \tag{57}$$

The total moduli space has the dimension  $30n - 8$ .

The Weierstrass equation can see only the partial information. On  $z = 0$ , we can see only the projected part for the divisor  $(r - 2t)|_r = (r - 2t) \cdot r = n - 2$ . This means that we cannot count the right degree of freedom. For example the instanton number is not  $(r + r_0 + 12t)|_r = 12 + n$ , as shown just before. Since the instanton number is  $8 + n$ , it seems that we can reproduce the same spectrum by redefining the instanton number  $8 + n \equiv 12 + n'$ . Comparing the spectrum in the previous subsection, indeed we reproduce the spectrum. Our present model should be viewed as one originating  $E_8$  located at  $(r - 2t)$ , whose moduli space is  $30n' + 112 = 30n - 8$ . The physics should be equivalent since it is completely specified by the instanton embedding. However, the relative relation between two  $E_8$ s are different. For example, putting the other  $E'_8$  on  $r_0$ , we have relative intersection number  $(r - 2t) \cdot r_0 = -2$  signaling the inconsistency. Also The gravitational anomaly cancellation seems difficult, however we can show always we can.

If we place the other  $E'_6 \times U(2)'$ , as a subgroup of  $E'_8$  on,  $(r_0 + 2t)$ , we have a similar spectrum

$$\frac{1}{6}(4 - n)(\mathbf{2}, \mathbf{27})_1 + \frac{1}{3}(26 - 2n)(\mathbf{1}, \mathbf{27})_{-2} + \frac{1}{2}(44 - 3n)(\mathbf{2}, \mathbf{1})_{-3}. \quad (58)$$

so the total moduli space is again  $(30n - 8) + (-30n + 232) = 224$ , again leading to gravitational anomaly cancellation. This condition can be tracked back that two  $E_8$  subgroups should be independent

$$(r - 2t) \cdot (r_0 + 2t) = 0.$$

Although formally anomalies cancel, it is consistent if all the coefficients are nonnegative. Since the coefficients of  $(\mathbf{2}, \mathbf{27})$  are of opposite signs, so only  $n = 4$  case seems valid. As we noted, this is the ‘parallel separation’ condition. It is also possible a different subgroup of  $E'_8$  can give the desired dimension of the moduli space  $-30n + 232$ . If we put all the hidden sector gauge group along the divisor  $(r_0 + 2t)$  this condition is valid. Since the total dimension of the moduli space does not change by spontaneous symmetry breaking, we can construct the subgroup in the hidden sector. Again, this shows that the gravitational anomaly cancellation comes from the embedding to  $E_8$ .

We can consider more general embedding of  $E_6 \times U(2)$  singularity in a form

$$D = 8(r + at) + 2(r + bt) + \dots \quad (59)$$

In case  $a \neq b$ , there is no easy argument that the model comes from  $E_8$ , since there can be nonzero intersection with a divisor supporting the hidden sector group, i.e. subgroup of the other  $E'_8$ . In other words, it is hard to construct the mirror model, where all the visible brane is disjoint from the hidden brane. Usually at least one of the intersections has a negative intersection number, and even if positive, it should be proportional to

the number of charged matter under both groups, weighted by group theoretical factors. If the group is not small enough, the total sum of the coefficients exceeds 24, which is required by the Calabi–Yau condition  $D = -12K_{B'}$ . However for such small group, the origin can also be tracked from the subgroup of  $SO(32)$ . With our method, it seems not possible to construct the model which is not the subgroup of  $E_8 \times E_8$ . At best one  $E_8$  can carry the sum of divisors  $12r + 2rt$ . We can state such condition as

$$12r + 24t - \sum_{i \in E_8 \text{ subgroup}} (\text{ord } \Delta_i) S_i \geq 0 \quad (60)$$

where the inequality means the divisor on LHS is effective. Also, if the total gauge symmetry lies outside  $E_8$  unified group, we have no physical reason for anomaly cancellation (31).

#### 4.4 $SO(10)$ and its extended groups

It suffices to study  $SO(10)$  for more general case. The splitness condition implies the relation [8]

$$g_{12+3n} = 2s_{4+n}^3, \quad f_{8+2n} = -3s_{4+n}^2, \quad g_{12+2n} = q_{6+n}^2 - f_{8+n}s_{4+n}, \quad (61)$$

so that

$$\Delta = 108z^7 s_{4+n}^3 q_{6+n}^2 + O(z^8). \quad (62)$$

As before for generic  $f_{8+n}$  we have only  $SO(10)$ , however for  $f_{8+n} \rightarrow 0$  we have gauge symmetry enhancement  $SO(10) \times U(2) \times U(1)$  without changing the leading order form (62). We have rank one symmetry enhancement by  $s_{4+n} \rightarrow 0$  to  $E_6$ . Instead if we send  $q_{6+n} \rightarrow 0$ , the singularity is generic  $D_6$  without splitting, which describes  $SO(11)$  gauge symmetry. Further splitness condition imposing inter-relations among  $g_{12+n}, f_{8+n}, s_{4+n}$  yields  $SO(12)$ .

The pure  $E_6$  theory cannot be a unification group, since then **10** is not possible from the branching of the adjoint **78**. Also from the form of equation, we have  $SO(11)$  enhancement direction, implying a larger symmetry. The structure of Weierstrass equation requires the  $E_8$  embedding.

In terms of divisors of  $\mathbb{F}_n$ , we have

$$\begin{aligned} F &= 2r &+& 2r + 4r_0 + 8t, \\ G &= 3r &+& 3r + 6r_0 + 12t, \\ D &= \underbrace{7r}_{D_5} &+& \underbrace{5r + 12r_0 + 24t}_{D'}. \end{aligned} \quad (63)$$

From the products  $r \cdot (2r + 4r_0 + 8t) = 2(n+4)$  and  $r \cdot (3r + 6r_0 + 12t) = 3(n+4)$  we draw out the above leading order relations of  $g_{12+3n}, f_{8+2n}$  have a special splitting condition.

Since they cancel out each other in the discriminant, we rely on the next leading order terms in  $z$ , which is that of  $E_6$  in the previous section. Thus we inherit the same number of  $\mathbf{10}$  of  $SO(10)$  from the branching of  $\mathbf{27}$  of  $E_6$ , whereas two  $\mathbf{16}$ 's of  $SO(10)$  is absorbed by Higgs mechanism.  $D'$  is decomposed into  $2(r + 6t) + 3(r + 4t)$ , up to irrelevant  $r_0$ 's. With respect to  $r$ , they have respectively  $(n + 6)$  and  $(n + 4)$  intersections, we have

$$(n + 6)\mathbf{10} + (n + 4)\mathbf{16} \quad (64)$$

localized along the corresponding intersections.

In the same way, we can calculate  $SO(10) \times U(3)$  spectrum. From the branching from  $E_8$ , we have no  $(\mathbf{10}, \mathbf{1})$ , so the only vector multiple under the  $SO(10)$  is  $(\mathbf{10}, \mathbf{3})$  whose number is then  $(n + 6)/3$ . We have

$$\frac{1}{12}(n - 12)(\mathbf{16}, \mathbf{3})_{-1} + \frac{1}{3}(n + 6)(\mathbf{10}, \mathbf{3})_{-2} + \frac{1}{3}(4n + 42)(\mathbf{1}, \mathbf{3})_{-4} + \frac{1}{4}(3n + 28)(\mathbf{16}, \mathbf{1})_3. \quad (65)$$

We can check that the matter curve is again  $(r + 8t)$ , since we set  $f_{8+n} = 0$  to have the desired symmetry enhancement. Indeed we verify (32)

$$\sum_{R,q} \frac{q^2}{V^2} \ell(R) n_{R,q} = 3(n + 8),$$

which is universal with respect  $SO(10) \times U(1)$  and  $SU(3) \times U(1)$ . In the heterotic side we use the vector  $V^2 = (2, 2, 2, 0^5)^2 = 12$ , so that the matter spectrum is also obtained by the index (28)  $n_q = q^2(12 + n)/12 - 2$ .

Finally, consider a semisimple group  $SO(10) \times U(2) \times U(1)$ . This is the common subgroup of  $E_6 \times U(2)$  and  $SO(10) \times U(3)$ . These enhanced groups are very useful in understanding the structure of the subgroup. In terms of divisors,

$$10r \longrightarrow \left\{ \begin{array}{l} 8r + 2r \\ 7r + 3r \end{array} \right\} \longrightarrow 7r + r + 2r. \quad (66)$$

From the branching of the adjoint of  $E_8$ , we have more than one matter charged under both  $SO(10)$  and  $SU(2)$ . We use the inheritance condition (29). Considering the enhanced groups  $E_6 \times SO(10)$  and  $SO(10) \times U(3)$  we have, for example,  $(\mathbf{2}, \mathbf{16})_1$  comes from the branching of  $(\mathbf{2}, \mathbf{27})_1$  of  $E_6 \times U(2)$ , and also from of  $(\mathbf{3}, \mathbf{16})_{-1}$  of  $SO(10) \times U(3)$ . So we have the multiplicity  $n_{(\mathbf{2}, \mathbf{16})} = \frac{1}{2}n_{(\mathbf{2}, \mathbf{27})} + \frac{1}{2}n_{(\mathbf{3}, \mathbf{16})}$ . Therefore we have highly nontrivial spectrum

$$\begin{aligned} & \frac{1}{8}(n - 4)(\mathbf{2}, \mathbf{16})_{1,1} + \frac{1}{4}(n + 4)(\mathbf{2}, \mathbf{10})_{1,2} + \frac{1}{12}(n - 12)(\mathbf{2}, \mathbf{1})_{1,0} \\ & + \frac{1}{24}(17n + 156)(\mathbf{1}, \mathbf{16})_{-2,3} + \frac{1}{2}(n + 8)(\mathbf{1}, \mathbf{10})_{-2,-2} + (10 + n)(\mathbf{1}, \mathbf{1})_{-2,4} \\ & + \frac{1}{12}(17n + 180)(\mathbf{2}, \mathbf{1})_{3,4} + \frac{1}{24}(n - 36)(\mathbf{1}, \mathbf{16})_{0,1}. \end{aligned}$$

At this stage, the spectrum seems not be realistic since  $n$  should be a multiple of 12 which is also larger than 36. A certain rearrangement of the divisor supporting the group, or blowing-up is necessary. In what follows, we are content with formal check of consistency conditions. The dimension of the moduli space is  $30n + 112$ . It satisfies GS condition

$$\ell(\mathbf{2})\ell(\mathbf{16})n_{(\mathbf{2},\mathbf{16})} + \ell(\mathbf{2})\ell(\mathbf{10})n_{(\mathbf{2},\mathbf{10})} = n,$$

and consistent with the Higgsing that leaving only  $SO(10)$

$$\begin{aligned} n_{\mathbf{16}} &= 2n_{(\mathbf{2},\mathbf{16})} + n_{(\mathbf{1},\mathbf{16})_{-2,3}} + n_{(\mathbf{1},\mathbf{16})_{0,1}} = n + 4, \\ n_{\mathbf{10}} &= 2n_{(\mathbf{2},\mathbf{10})} + n_{(\mathbf{1},\mathbf{10})} = n + 6. \end{aligned}$$

We can check the matter curve relation (32) is universal for any of four combinations between  $SO(10)$ ,  $SU(2)$  and  $U(1)$ ,  $U(1)'$ .

This group is obtained from the above  $SO(10)$  by making  $f_{8+n} = 0, g_{12+n} = 0$ . The resulting discriminant is

$$\Delta = z^7 \left[ (-36f_8^2 s_{4+n}^2 z^3 + 54(g_{12}q_{6+n}^2 z^2 + 2g_{12}s_{4+n}^3 z + 2f_8 s_{4+n}^4))z + 27q_{6+n}^4 z + 4s_{4+n}^3 q_{6+n}^2 \right] \quad (67)$$

In terms of  $f_8 \rightarrow \infty$  and  $g_{12} \rightarrow \infty$  with  $f_8^3/g_{12}^2$  fixed, we find a hierarchy in orders  $O(f_8^2)$  and  $O(f_8) \sim O(g_{12})$ , therefore we have the factorization  $\Delta \sim z^7 \cdot z^2 \cdot z$ .

## 4.5 $F_4$

We describe a non-simply laced group  $F_4$ , which is described by *generic*  $E_6$ . So it carries the same orders as  $E_6$ ,

$$\begin{aligned} F &= 3r + \underbrace{r + 4r_0 + 8t}_{F'} \\ G &= 4r + \underbrace{2r + 6r_0 + 12t}_{G'}, \\ D &= 8 \underbrace{r}_{F_4=E_6} + \underbrace{4r + 10r_0 + 20t}_{2D'_{26}} + \underbrace{2r_0 + 4t}_{D'_1}. \end{aligned} \quad (68)$$

The non-splitness condition tells us that  $D'$  is not proportional to  $G'$ , so it should be decomposed. As always there is a unimportant ambiguity in the distribution of  $r_0$ . We have only splitness 2 from Tate's algorithm. The product of  $F_4$  divisor and  $D'$  shows we have  $2 \cdot (2n + 10)$  intersections.

It seems not possible to deform the Weierstrass equation into the desirable form, because the Weierstrass equation shows the singularity form when the *entire* the subgroups of  $E_8$  is lying on the original position  $r$ . We note that the next order term plays a role. We see  $f_4$  originated from the tuning  $f_4 \cdot g_{12+2n} = f_{8+n}^2$  agreeing with  $D'_1 = (2F' - G')$



and  $r \cdot D'_1 = 4$ . Therefore we have  $(2n + 10)/2 = (n + 5)$  matter representations in **26** and four **1s**. A similar thing happens for a *generic*  $A_1$  singularity in Ref. [8], where we have ‘antisymmetric’ representation **1** of  $SU(2)$  with multiplicity  $(4 + 2n)/2 = n + 2$ . Even if we have difficulty in expressing a generic group with monodromy reduction, we can calculate the multiplicity from the intersection theory.

## 5 Conclusion

We have illustrated how to specify a gauge theory and obtain its matter spectrum in F-theory. The essential problem is how to decompose the discriminant locus, in (4) and (44). It is analogous to configuring D7-branes in the internal manifold. Duality to heterotic string (7) suggests that our manifold should be compatible to both elliptic and K3 fibration. This limits the possible gauge groups as ones in heterotic string,  $E_8 \times E_8$  or  $SO(32)$ , from independent considerations of Weierstrass equation or the conservation of 7-brane charges. Most of vacua admits an interpretation that they are obtained by symmetry breaking of the unified group. Also, several constraints of model building, such as anomaly cancellation conditions from Green–Schwarz mechanism, the formation of matter branes and the dimension of moduli space, indicate the unification towards the above unification group. This unification condition is only evaded by blowing-up in the base of elliptic fibration and including the resulting exceptional divisors in the discriminant locus. Even in this case, the above unification group is a good starting point to consider top-down approach. Usually by top-down approach, we meet many unwanted charged matters, as well as hidden sectors used for symmetry breaking.

A matter curve comes from the intersection between 7-branes. If we have explicit information on 7-branes, not one for local unification group but the complete set of 7-branes in the theory, we can obtain the matter curve without ambiguity. To see this we considered semisimple gauge group. In Section 4, we took examples having semisimple gauge group. In every case, spectrum and moduli space matches perfectly to that of  $E_8 \times E_8$  in both F-theory (geometric) and heterotic (gauge bundle) side. In the latter, we can calculate the spectrum using index theorem, whose vacuum is parameterized by spectral cover.

The requirements from Green–Schwarz mechanism (22)-(25) shows that, in the expansion of the intersection of two 7-branes, each matter curve (as a divisor in 7-brane support) is weighted by a group theoretical factor depending on the localized matter. The nontriviality comes if we embed the groups in the exceptional group, since we have no notion of parallel separation of exceptional branes, unlike that of D-branes. Thus we cannot determine the spectrum solely by geometric data of the intersection. We obtained the following rule, from the corresponding heterotic dual model in Subsec. 3.2; To completely specify the matter spectrum, we should consider every possible enhanced gauge groups;

the matter multiplicity is equally inherited from those enhanced group. Thus knowing the global structure is again important.

We also have 7-branes responsible for  $U(1)$ , which do not directly give the matter multiplicity, as in (32). The anomaly constraint confirms the existence of such  $U(1)$  brane. Such  $U(1)$ 's provided additional constraints on the matter coupling (implicitly used in [34]).

In this way we can find many nontrivial vacua of F-theory with semisimple group, opening up more possibility for model building. They admit heterotic duals with line bundle backgrounds, some of which are close to many models suggested so far. An additional group outside the conventional unification group of  $SU(5)$  or  $SO(10)$  can also play a role, either providing one source of the interaction of the Standard Model in the larger unification group, or supplementing additional constraints.

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group	$R$	multiplicity	$\ell(R)$	$s_R$	$x_R$	$y_R$
$E_7$	<b>133</b>	1	36	$\cdot$	0	24
	$\frac{1}{2}\mathbf{56}$	$8 + n$	$\frac{1}{2}12$	3	0	$\frac{1}{2}6$
$E_6$	<b>78</b>	1	24	$\cdot$	0	18
	<b>27</b>	$6 + n$	6	4	0	3
$SO(10)$	<b>45</b>	1	16	$\cdot$	4	12
	<b>16</b>	$4 + n$	4	3	-2	3
	<b>10</b>	$6 + n$	2	2	2	0
$SO(2N)$	$\mathbf{N(2N-1)}$	1	$4N - 4$	$\cdot$	$4N - 16$	12
	$\mathbf{2^{N-2}}$	$\cdot$	$2^{N-3}$	$\cdot$	$-2^{N-4}$	$3 \cdot 2^{N-5}$
	$\mathbf{2N}$	$\cdot$	$N - 1$	$\cdot$	2	0
$SU(5)$	<b>24</b>	1	10	$\cdot$	10	6
	<b>10</b>	$2 + n$	3	4	-3	3
	<b>5</b>	$16 + 3n$	1	1	1	0
$SU(N)$	$\mathbf{N^2 - 1}$	1	$2N$	$\cdot$	$2N$	6
	$\mathbf{N(N-1)/2}$	$\cdot$	$N - 2$	$\cdot$	$N - 8$	$N - 2$
	$\mathbf{N}$	$\cdot$	1	1	1	0

Table 2: Some group invariants, absolutely normalized so that the trace is the same for equivalent representations before and after branching.  $s_R$  is the splitness of  $E_8$  embedding.